New Octonionic Representations of E_6 and E_7

Tevian Dray
(joint work with Corinne Manogue and Robert Wilson)

Department of Mathematics
Oregon State University
http://www.math.oregonstate.edu/~tevian

Oregon State University

(supported by FQXi and the John Templeton Foundation)

With thanks to:

- Rob Wilson, who showed us how to get to E_8;
- David Fairlie & Tony Sudbery, who got us started in the 1980s, Paul Davies, who believed in us from the start, David Griffiths, who taught us physics (and math), and Jim Wheeler, who explained the conformal group to us;
- Jörg Schray (Ph.D. 1994), Jason Janesky (1997–1998), Aaron Wangberg (Ph.D. 2007), Henry Gillow-Wiles (M.S. 2008), Joshua Kinkaid (M.S. 2012), Lida Bentz (M.S. 2017), and Alex Putnam (M.S. 2017), who taught us as much as we taught them;
- John Huerta and Susumu Okubo, who helped along the way;
- and FQXi, the John Templeton Foundation, and the Institute for Advanced Study for financial support.

Division Algebras

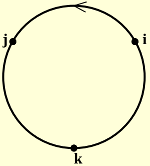
Real Numbers

$$\mathbb{R}$$

Quaternions

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$$

$$q = (x + yi) + (r + si)j$$



Complex Numbers

$$\mathbb{C} = \mathbb{R} \oplus \mathbb{R}i$$

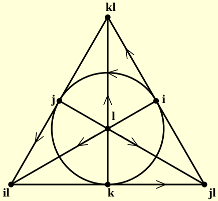
$$z = x + yi$$

Octonions

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$$

Split Octonions

$$\mathbb{O}' = \mathbb{H} \oplus \mathbb{H}L$$



$$I^2 = J^2 = -U, \; L^2 = +U$$

Split Division Algebras

$$I^2 = J^2 = -U,\ L^2 = +U$$

Signature $(4,4)$:

$$x = x_1 U + x_2 I + x_3 J + x_4 K + x_5 KL + x_6 JL + x_7 IL + x_8 L \Longrightarrow$$

$$|x|^2 = x\bar{x} = (x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_5^2 + x_6^2 + x_7^2 + x_8^2)$$

Null elements:

$$|U \pm L|^2 = 0$$

Projections:

$$\left(\frac{U \pm L}{2}\right)^2 = \frac{U \pm L}{2}$$

$$(U + L)(U - L) = 0$$

The Freudenthal–Tits Magic Square

Freudenthal (1964), Tits (1966):

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)}$
$\mathbb{H}'$	$\mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3,\mathbb{C})$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

- $\mathfrak{su}(3,\mathbb{K}'\otimes\mathbb{K})$, generated by *anti-Hermitian* matrices.
  ($p\in\mathbb{K}'\otimes\mathbb{K}$ and $q\in\mathrm{Im}\mathbb{K}+\mathrm{Im}\mathbb{K}'$)

$$D_q=\begin{pmatrix}q&0&0\\0&-q&0\\0&0&0\end{pmatrix},\quad S_q=\begin{pmatrix}q&0&0\\0&q&0\\0&0&-2q\end{pmatrix},\quad X_p=\begin{pmatrix}0&p&0\\-\bar{p}&0&0\\0&0&0\end{pmatrix}$$

$$Y_p=\begin{pmatrix}0&0&0\\0&0&p\\0&-\bar{p}&0\end{pmatrix},\quad Z_p=\begin{pmatrix}0&0&-\bar{p}\\0&0&0\\p&0&0\end{pmatrix}$$

Albert Algebra

Albert algebra: 3×3 *Hermitian* matrices $\mathcal{A}$ over $\mathbb{O}$.

Jordan product:

$$\mathcal{X} \circ \mathcal{Y} = \frac{1}{2}(\mathcal{X}\mathcal{Y} + \mathcal{Y}\mathcal{X})$$

Freudenthal product:

$$\mathcal{X} * \mathcal{Y} = \mathcal{X} \circ \mathcal{Y} - \frac{1}{2}\Big((\mathrm{tr}\mathcal{X})\,\mathcal{Y} + (\mathrm{tr}\mathcal{Y})\,\mathcal{X}\Big) + \frac{1}{2}\Big((\mathrm{tr}\mathcal{X})(\mathrm{tr}\mathcal{Y}) - \mathrm{tr}(\mathcal{X} \circ \mathcal{Y})\Big)\,\mathcal{I}$$

Determinant:

$$\det(\mathcal{X}) = \frac{1}{3}\,\mathrm{tr}\Big((\mathcal{X} * \mathcal{X}) \circ \mathcal{X}\Big)$$

Idea: $\mathrm{tr}(\mathcal{X} \circ \mathcal{Y}) \longleftrightarrow \mathcal{X} \cdot \mathcal{Y}, \quad \mathcal{X} * \mathcal{Y} \longleftrightarrow \mathcal{X} \times \mathcal{Y}$

$\mathfrak{so}(\mathbf{p},\mathbf{q}) \subset \mathfrak{so}(\mathbf{p}+1,\mathbf{q}+1)$

$$\begin{pmatrix} \mathfrak{so}(p,q) & v & w \\ -v^\dagger & \multicolumn{2}{c}{\multirow{2}{*}{$\mathfrak{so}(1,1)$}} \\ -w^\dagger & & \end{pmatrix}$$

"Conformalization:"

$$\mathfrak{so}(p+1,q+1) = \mathfrak{so}(p,q) \oplus 2 \times (\mathbf{p}+\mathbf{q}) \oplus \mathfrak{so}(1,1)$$

- $\mathfrak{so}(p+1,q+1)$ contains $\mathfrak{so}(p,q)$ *and* two $p+q$ vectors.

Conformal group (e.g. $p=3$, $q=1$)

$=$ Lorentz group $+$ translations $+$ conformal translations $+$ dilation

Decomposing $\mathfrak{e}_8$ over $\mathfrak{e}_6$

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- The 6 of $\mathfrak{sl}(3, \mathbb{R})$ are "color labels": $\{I \pm IL, J \pm JL, K \pm KL\}$.

[Dray, Manogue, Wilson (2023): A New Division Algebra Representation of E_6]

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- Over $\mathbb{O}$, $(K \pm KL)\mathcal{I}$ is nested; really $\sim G_{K \pm KL} \in \mathfrak{g}_2'$.

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Freudenthal Description of $\mathfrak{e}_7$

$$\Theta = (\phi, \rho, \mathcal{A}, \mathcal{B}) \in \mathfrak{e}_7$$
$$\boldsymbol{\mathcal{P}} = (\mathcal{X}, \mathcal{Y}, p, q) \in \mathbf{56}$$

$\phi \in \mathfrak{e}_6$, $\rho \in \mathbb{R}$, $\mathcal{A}, \mathcal{B} \in \mathrm{H}_3(\mathbb{O})$, $\mathcal{X}, \mathcal{Y} \in \mathrm{H}_3(\mathbb{O})$, $p, q \in \mathbb{R}$

$$\mathcal{X} \longmapsto \phi(\mathcal{X}) + \frac{1}{3}\rho\,\mathcal{X} + 2\mathcal{B} * \mathcal{Y} + \mathcal{A}\,q$$
$$\mathcal{Y} \longmapsto 2\mathcal{A} * \mathcal{X} + \phi'(\mathcal{Y}) - \frac{1}{3}\rho\,\mathcal{Y} + \mathcal{B}\,p$$
$$p \longmapsto \mathrm{tr}(\mathcal{A} \circ \mathcal{Y}) - \rho\,p$$
$$q \longmapsto \mathrm{tr}(\mathcal{B} \circ \mathcal{X}) + \rho\,q$$

Decomposing $\mathfrak{e}_8$ over $\mathfrak{e}_7$

- $\mathfrak{e}_8 = \mathfrak{e}_7 \oplus 2 \times \mathbf{56} \oplus \mathfrak{su}(2)$
- $\mathfrak{e}_7$ is the conformalization of $\mathfrak{e}_6$:
  $\mathfrak{e}_6$, two Albert algebras, and a dilation.
- Each **56** is a minimal representation of $\mathfrak{e}_7$,
  generated by two Albert algebras and two scalars.
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- The action of $\mathfrak{e}_7$ on **56** uses the Freudenthal product and the trace of the Jordan product.

$\Longrightarrow$ These products *must* be realized as commutators in $\mathfrak{e}_8$!!

Freudenthal Towers

$$\begin{array}{ccc} & \mathfrak{e}_{6(-26)} & \\ / & & \backslash \\ K_-\mathcal{A} & & K_+\mathcal{B} \\ \backslash & & / \\ & \mathfrak{so}(1,1) & \end{array}$$

$$\begin{array}{cc} pA_{I_+} & pA_{J_-} \\ | & | \\ J_+\mathcal{Y} & I_+\mathcal{Y} \\ | & | \\ I_-\mathcal{X} & J_-\mathcal{X} \\ | & | \\ qA_{J_+} & qA_{I_-} \end{array}$$

$$\mathfrak{sl}(2,\mathbb{R})$$

$$\mathfrak{e}_{8(-24)} = \mathfrak{e}_{7(-25)} \oplus 2 \times \mathbf{56} \oplus \mathfrak{sl}(2,\mathbb{R})$$

$$K_\pm = \frac{1}{2}(K \pm KL) \ \ldots$$

Two Subalgebras of $\mathbb{O}'$

$$\{I \pm IL, J \pm JL, K \mp KL\} \subset \mathbb{O}'$$

- These are 3-dimensional *subalgebras*!
- The only nonzero product is $(I \pm IL)(J \pm JL) = 2(K \mp KL)$.

Albert Algebra as Commutators

“Dot”:

$$[(K \pm KL)\mathcal{X}, (I \mp IL)\mathcal{Y}] = \mathrm{tr}(\mathcal{X} \circ \mathcal{Y})\, A_{J \pm JL}$$

“Cross”:

$$[(I \pm IL)\mathcal{X}, (J \pm JL)\mathcal{Y}] = 4\,(K \mp KL)\,\mathcal{X} * \mathcal{Y}$$

Determinant:

$$\big[K_\pm \mathcal{A}, [I_\pm \mathcal{A}, J_\pm \mathcal{A}]\big] = \mp(\det \mathcal{A})\, G_L$$

[Dray, Manogue, Wilson (2023): A New Division Algebra Representation of E_7]

Graded Lie algebras

$$\mathfrak{g} = \mathfrak{g}_{-m} \oplus \dots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_m$$

- $\mathfrak{g}_0$ semisimple
- $\mathfrak{g}_p$ nilpotent for $p \neq 0$
- $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$

Gradings of Exceptional Lie Algebras

$$\mathfrak{so}(p+1, q+1) = (\mathbf{p}+\mathbf{q}) \oplus \big(\mathfrak{so}(p,q) \oplus \mathfrak{so}(1,1)\big) \oplus (\mathbf{p}+\mathbf{q})$$

$$\begin{aligned}
\mathfrak{e}_{7(-25)} &= \mathbf{27} \oplus \big(\mathfrak{e}_{6(-26)} \oplus \mathfrak{so}(1,1)\big) \oplus \mathbf{27} \\
\mathfrak{e}_{8(-24)} &= \mathbf{56} \oplus \big(\mathfrak{e}_{7(-25)} \oplus \mathfrak{sl}(2,\mathbb{R})\big) \oplus \mathbf{56} \\
&= (2 \times \mathbf{1}) \oplus \mathbf{27} \oplus (2 \times \mathbf{27}) \\
&\qquad \oplus \big(\mathfrak{e}_{6(-26)} \oplus \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(1,1)\big) \\
&\qquad \oplus (2 \times \mathbf{27}) \oplus \mathbf{27} \oplus (2 \times \mathbf{1})
\end{aligned}$$

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{0} \oplus \mathfrak{g}_{1} \oplus \mathfrak{g}_{2} \oplus \mathfrak{g}_{3}$$

SUMMARY

Albert algebras $\subset \mathfrak{e}_8$

- Wilson, Dray, and Manogue: An octonionic construction of E_8 ..., Innov. Incidence Geom. **20**, 611–634 (2023); arXiv.org:2204.04996
- Dray, Manogue, and Wilson: A New ... Representation of E_6; arXiv.org:2309.00078
- Dray, Manogue, and Wilson: A New ... Representation of E_7; (in preparation)