ENDS OF MANIFOLDS: RECENT PROGRESS

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Abstract. In this note we describe some recent work on ends of manifolds. In particular, we discuss progress on two different approaches to generalizing Siebenmann's thesis to include manifolds with non-stable fundamental groups at infinity.

1. Introduction

In this note we discuss some of our recent work on ends of manifolds. For simplicity we focus our attention on one-ended open manifolds.

- A manifold M^n is *open* if it is noncompact and has no boundary.
- A subset V of M^n is a *neighborhood of infinity* if $\overline{M^n - V}$ is compact.
- M^n is *one-ended* if each neighborhood of infinity contains a connected neighborhood of infinity.

Example 1. *$\mathbb{R}^n$ is an open n-manifold for all $n \geq 1$. If $n \geq 2$, then $\mathbb{R}^n$ is one-ended.*

Example 2. *If P^n is a closed connected manifold, then $P^n \times \mathbb{R}^k$ is an open manifold for all $n \geq 1$. $P^n \times \mathbb{R}^k$ is one-ended iff $k \geq 2$.*

Example 3. *Let P^n be a compact manifold with non-empty connected boundary. Then int (P^n) is a one-ended open manifold.*

A natural question to ask about open manifolds is the following.

Question. When is an open n-manifold M^n just the interior of a compact manifold with boundary?

Equivalent Question. When does M^n contain an "open collar" neighborhood of infinity? (V is an *open collar* if $V \approx \partial V \times [0, 1)$.

These questions were answered (in high dimensions) by Siebenmann in his 1965 Ph.D. thesis.

Theorem 1.1 (see [Si])**.** *A one ended open n-manifold M^n ($n \geq 6$) contains an open collar neighborhood of infinity if and only if each of the following is satisfied:*

(1) *M^n is inward tame at infinity,*
(2) *π_1 is stable at infinity, and*

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(3) $\sigma_\infty(M^n) \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\varepsilon(M^n))])$ *is trivial.*

In the above theorem:

- *inward tame* means $\forall$ neighborhood V of infinity, $\exists$ homotopy $H : V \times [0,1] \to V$ such that $H_0 = id$ and $\overline{H_1(V)}$ is compact.
- π_1 *stable at infinity* means $\exists$ a sequence $V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$ of neighborhoods of infinity with $\bigcap V_i = \emptyset$ and inclusion induced homomorphisms all isomorphisms:

$$\pi_1(V_0) \xleftarrow{\lambda_1} \pi_1(V_1) \xleftarrow{\lambda_2} \pi_1(V_2) \xleftarrow{\lambda_3} \cdots .$$

- Condition 3) ensures that the V_i's have finite homotopy type.

Remark. Siebenmann's Theorem (and variations due to Quinn) have been extremely important in manifold topology—especially embedding theory. In other situations—for example the study of universal covering spaces—the hypotheses are too strong. Thus, it has been asked:

Question. Are there versions of Siebenmann's Theorem that apply to a more general class of manifolds?

The main goal of this note is to discuss two different (but related) programs for generalizing Siebenmann's Theorem.

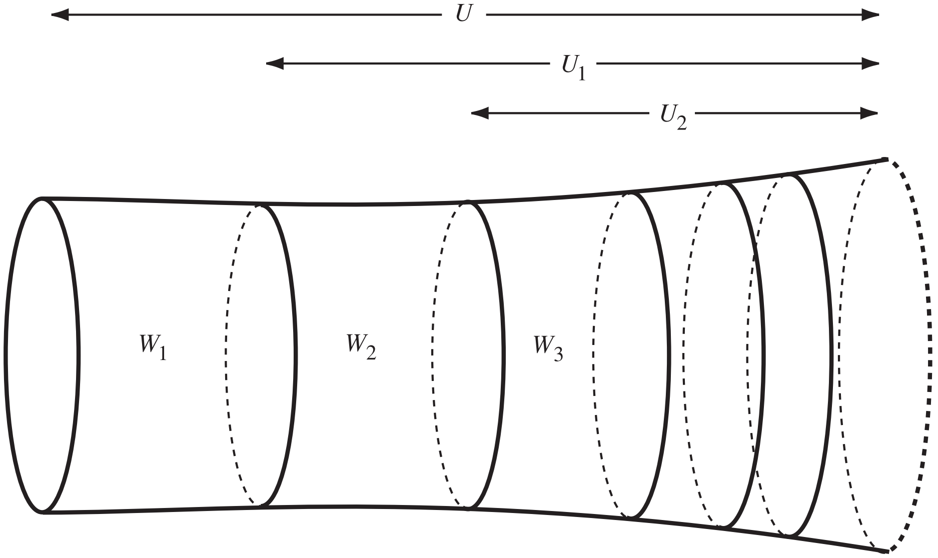
2. Generalizing Siebenmann's Thesis: Approach #1

We begin by generalizing the notion of an open collar to that of a "pseudo-collar". We then seek conditions that imply that a given open n-manifold contains a pseudo-collar neighborhood of infinity.

- A manifold U with compact boundary is a *homotopy collar* if $\partial U \hookrightarrow U$ is a homotopy equivalence.
- If, in addition, U contains arbitrarily small homotopy collar neighborhoods of infinity, we call U a *pseudo-collar.*

One nice aspect of a pseudo-collar structure is that it may be decomposed into a countable union of compact "one-sided h-cobordisms". (A *one-sided h-cobordism* can be deformation retracted onto one of its boundary components, but not necessarily onto the other.) These cobordisms have been the object of frequent study. See for example [DT] and Sections 11.1 and 11.2 of [FQ]. Given a pseudo-collar U and a cofinal sequence $U_1 \supset U_2 \supset U_3 \supset \cdots$ of homotopy collar neighborhoods of infinity, let $W_i = U_{i-1} - int(U_i)$. Then each $(W_i, \partial U_{i-1}, \partial U_i)$ is a one-sided h-cobordism. See Figure 1 for a schematic picture.

Example 4. *A particularly interesting collection of pseudo-collarable (but not collarable) open n-manifolds are the exotic universal covering spaces constructed by M. Davis in* [Da].



So far, our best theorem for ensuring pseudo-collarability in an open n-manifold is:

Theorem 2.1 (see [Gu1]). *A one-ended open n-manifold M^n ($n \geq 7$) is pseudo-collarable provided each of the following is satisfied:*

(1) *M^n is inward tame at infinity,*
(2) *π_1 is perfectly semistable at infinity,*
(3) *$\sigma_\infty(M^n) \in \varprojlim \left\{ \widetilde{K}_0\pi_1(M^n \backslash A) \mid A \overset{cpt.}{\subset} M^n \right\}$ is zero, and*
(4) *π_2 is semistable at infinity.*

In this theorem:

- π_1 *semistable at infinity* means $\exists$ a sequence $V_0 \supseteq V_1 \supseteq V_2 \supseteq \cdots$ of neighborhoods of infinity with $\bigcap V_i = \emptyset$ and inclusion induced homomorphisms all surjective:
$$\pi_1(V_0) \xleftarrow{\lambda_1} \pi_1(V_1) \xleftarrow{\lambda_2} \pi_1(V_2) \xleftarrow{\lambda_3} \cdots .$$
- *perfectly semistable* means that, in addition, we can arrange that $\ker(\lambda_i)$ is perfect for each i,
- requiring that $\sigma_\infty(M^n) = 0$ ensures that the V_i's have finite homotopy types, and
- π_2 *semistable at infinity* means what you think it does...

Remark. Conditions 1)-3) are also necessary. Hence, the following question is natural.

Question. Can Condition 4) be eliminated from Theorem 2.1?

Another intriguing open problem is:

Question. Does condition 2) follow from Condition 1)?

Combining the above two questions we arrive at:

Big Question #1. Do conditions 1) and 3) suffice?

3. Generalizing Siebenmann's Thesis: Approach #2

Instead of viewing Theorem 1.1 as detecting open collar neighborhoods of infinity, one may view it as answering the question: "When can an open manifold be compactified to a manifold with boundary by adding a boundary $(n-1)$-manifold?". Taking this point of view, our second approach to generalizing Theorem 1.1 is to look for compactifications which permit a less rigid sort of boundary (a "$\mathcal{Z}$-boundary").

- A closed subset A of a compact ANR Y is a *$\mathcal{Z}$-set* if, for every open set U of Y, $U\backslash A \hookrightarrow U$ is a homotopy equivalence.
- A compactification $\widehat{X}$ of a space X is a *$\mathcal{Z}$-compactification* if $\widehat{X}\backslash X$ is a $\mathcal{Z}$-set in $\widehat{X}$. In this case, we call $\widehat{X}\backslash X$ a *$\mathcal{Z}$-boundary* for X.

Example 5. *If P^n is a manifold with boundary, then any closed subset of ∂P^n is a $\mathcal{Z}$-set in P^n.*

Example 6. *Adding a manifold boundary to an open manifold is a (particularly nice) $\mathcal{Z}$-compactification.*

Example 7. *Davis' exotic universal covering spaces admit $\mathcal{Z}$-compactifications—but not manifold compactifications.*

Example 8. *If P^n is a closed aspherical manifold with CAT(0) or word hyperbolic fundamental group, then $\widetilde{P}^n$ admits a $\mathcal{Z}$-compactification.*

Question. Under what conditions does a one-ended open manifold admit a $\mathcal{Z}$-compactification?

For the case of Hilbert cube manifolds, this question was answered by the following theorem.

Theorem 3.1 (see CS)**.** *A Hilbert cube manifold X admits a $\mathcal{Z}$-compactification iff each of the following is satisfied.*

a) *X is inward tame at infinity.*

b) $\sigma_\infty(X) \in \varprojlim \left\{ \widetilde{K}_0\pi_1(X\backslash A) \mid A \overset{cpt.}{\subset} X \right\}$ *is zero.*

c) $\tau_\infty(X) \in \varprojlim^1 \left\{ Wh\pi_1(X\backslash A) \mid A \overset{cpt.}{\subset} X \right\}$ *is zero.*

Corollary 3.2. *For any locally compact ANR Y, the above conditions are necessary and sufficient for $Y \times [0,1]^\infty$ to be $\mathcal{Z}$-compactifiable.*

This corollary raises a natural question first posed by Chapman and Siebenmann.

Question. Are these conditions sufficient for the ANR Y itself to be $\mathcal{Z}$-compactifiable?

In [Gu2] we answered this question in the negative. The counterexample is a 2-dimensional polyhedron, but not a manifold. We consider the following to be an important open problem.

Big Question #2. Does an open manifold satisfying conditions a)-c) admit a $\mathcal{Z}$-compactification?

4. Recent Progress

In this section we describe some recent progress on some of the questions raised in the previous two sections. Proofs will be contained in a pair of papers that are currently in progress.

The first new result reveals a connection between conditions 1) and 2) of Theorem 2.1. Note, however, that it does not imply that condition 1) implies condition 2).

Theorem 4.1. *Let M^n be a one-ended open n-manifold. If M^n is inward tame at infinity, then π_1 is semistable at infinity.*

The second new result is related to "Big Question #2". Although it does not settle the problem, it provides the best possible "stabilized" answer to that question.

Theorem 4.2. *Let M^n be a one-ended open n-manifold ($n \geq 5$). Then $M^n \times [0,1]$ admits a $\mathcal{Z}$-compactification (in fact $M^n \times [0,1]$ is a "missing boundary manifold") if and only if M^n satisfies a)-c) of Theorem 3.1.*

Remark. In [Fe], Ferry has shown that if a k-dimensional polyhedron K satisfies a)-c), then $K \times [0,1]^{2k+5}$ admits a $\mathcal{Z}$-compactification. Previously, in [OB], O'Brien showed that $[0,1]^3$ suffices if K is a one-ended open manifold.

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