# Polynomial Chaos Expansions for Random Ordinary Differential Equations

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**Abstract**

We consider numerical methods for finding approximate solutions to Ordinary Differential Equations (ODEs) with parameters distributed with some probability by the Generalized Polynomial Chaos (GPC) approach. In particular, we consider those with forcing functions that have a random parameter in both the scalar and vector case. We then consider linear systems of ODEs with deterministic forcing and randomness in the matrix of the systems and conclude with a method of approximating solutions to the case where the system involves a nonlinear function of a matrix and a random variable.

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# 1 Introduction

There are many instances where one desires to model a physical system where heterogeneous micro-scale structures are present. One example of this is found in the modeling of movement of matter through porous soils with spatially dependent soil properties [4]. Another example is the propagation of electromagnetic waves through a dielectric material with variability in the relaxation time [2]. A major challenge in modeling such systems is that the parameters of the model vary according to the heterogeneities present. Thus any fixed value of the parameter used in the model is accompanied by an uncertainty for that value. One approach to dealing with this uncertainty is to use statistical sampling to approximate the expected values of the model parameters and to use these expected values in a deterministic model. The simulations needed for such an approach, however, can be expensive. Another approach to developing models that include uncertainty is by modeling the uncertain parameters with random variables from some probability distribution. One such approach called Polynomial Chaos was pioneered by Wiener and has been extended in recent years to the Generalized Polynomial Chaos (GPC) approach by Xiu and Karniadakis [9].

In the present paper we discuss the application of the GPC approach to models involving Ordinary Differential Equations (ODEs) with random parameters. We develop methods for approximating solutions to several types of these models including an extension of the work done in dealing with systems of ODEs with random forcing functions in [7]. We conclude the paper with a method of approximating the solution of a system of ODEs involving a nonlinear function that depends both on a random variable and a deterministic matrix.

# 2 Preliminaries: Polynomial Chaos, Orthogonal Polynomials, and Random Variables

Here we present a basic overview of several results from the literature on orthogonal polynomials, polynomial chaos, and random variables. The discussion here is intended to serve as a reference for the reader throughout the paper. Those interested in pursuing these topics further or more rigorously will be aided by the works listed in the References.

## 2.1 Orthogonal Polynomials

A **system of orthogonal polynomials** is a set of polynomials $\{\varphi_n\}_{n\in\mathcal{N}}$ with $\mathcal{N} \subset \mathbb{N}$ and $\deg(\varphi_n) = n$ that are orthogonal over a domain $S$ with respect to a positive measure $W$ [8]. That is for each $n, m \in \mathcal{N}$ we require

$$\int_S \varphi_n(x)\varphi_m(x)\, dW(x) = \gamma_n^2 \delta_{m,n}$$

where $\delta_{m,n}$ is the Kronecker delta

$$\delta_{m,n} := \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \tag{1}$$

and $\gamma_n$ is a real number called the **weight** of the polynomial $\varphi_n$. For the purposes of this paper we make some simplifying assumptions. We assume the domain $S$ is a subset of $\mathbb{R}$ and that $dW(x) = W(x)\, dx$ for all $x$ belonging to $S$ (the slight abuse of notation will cause no difficulties) for some function $W$, which we call the **weight function**. We also assume that our system of orthogonal polynomials is indexed by the natural numbers ($\mathcal{N}$=$\mathbb{N}$). With these assumption we define the **inner product** of two polynomials $\varphi_n$ and $\varphi_m$ as

$$\langle \varphi_n, \varphi_m \rangle := \int_S \varphi_n(x)\varphi_m(x)W(x)\, dx. \tag{2}$$

This provides us with the alternate characterization of orthogonality that for each $n, m \in \mathcal{N}$ we require

$$\langle \varphi_n, \varphi_m \rangle = \gamma_n^2 \delta_{m,n}.$$

And this tells us how to determine each weight $\gamma_n$

$$\gamma_n = \sqrt{\langle \varphi_n, \varphi_n \rangle}.$$

Another characterization of orthogonal polynomials is by their **basic recursion relation** [8]

$$\xi\varphi_n(\xi) = a_n\varphi_{n-1}(\xi) + b_n\varphi_n(\xi) + c_n\varphi_{n+1}(\xi) \tag{3}$$

where the real numbers $a_n$, $b_n$, and $c_n$ are called the **basic recursion coefficients** of the system of orthogonal polynomials. The term "basic" in the definitions of the recursion relation and coefficients is non-standard, and we use it here to emphasize the difference between these and similar terms appearing later in this paper. We now show how to make use of orthogonal polynomials by using them as a basis for expanding functions of random variables.

## 2.2 Generalized Polynomial Chaos Expansions

Suppose $\xi$ is a random variable from some known distribution on $S$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ is a complete system of orthogonal polynomial functions of $\xi$. Suppose also that $u\colon \Omega \to \mathbb{R}$, where $\Omega = [0,\infty) \times S$. If for each $t$ from $[0,\infty)$, $u(t,\cdot)$ is an $L^2(S)$ function, then $u$ admits of the following Weiner-Askey **Generalized Polynomial Chaos (GPC)** expansion on $\Omega$

$$u(t,\xi) = \sum_{n=0}^{\infty} u_n(t)\varphi_n(\xi) \tag{4}$$

where the convergence for each $t$ is in the $L^2(S)$ sense [8, 9]. The coefficient functions $u_n$ are called the **modes** of the expansion and are given by

$$u_n(t) := \frac{\langle u, \varphi_n\rangle}{\gamma_n^2}. \tag{5}$$

The **expected value** $\mu_u$ and the **variance** $\sigma_u^2$ of $u$ are given by [1]

$$\begin{aligned} \mu_u(t) &= u_0(t), \\ \sigma_u^2(t) &= \sum_{n=0}^{\infty} \gamma_n^2 u_n(t). \end{aligned} \tag{6}$$

We now give details for the distributions and systems of orthogonal polynomials that we consider in this paper.

## 2.3 The Beta Distribution and Jacobi Polynomials

In what follows we state results for a specific distribution and system of polynomials. Although these results apply more generally to several different distributions and their corresponding orthogonal systems of polynomials, we focus on a specific case for the sake of concreteness. For our purposes we assume the random variable $\xi$ comes from the **Beta distribution** on $[-1, 1]$ with shape parameters $a$ and $b$. This assumption suffices for the cases we are interested in here, since we are considering random variables distributed on some closed interval $[c, d]$, which we bijectively map to $[-1, 1]$. Assuming we begin with $\omega$ distributed on $[c, d]$, these maps are

$$\begin{aligned} \xi &= \frac{c+d}{2} + \omega\frac{d-c}{2}, \\ \omega &= \xi\frac{2}{d-c} - \frac{d+c}{d-c}. \end{aligned} \tag{7}$$

We also may refer to $\omega$ as the result of a shifting by $m$ and a scaling by $r$ of a random variable $\xi$. That is

$$\omega = r\xi + m. \tag{8}$$

We indicate that the random variable $\xi$ belongs to the Beta distribution with shape parameters $a$ and $b$ with the notation

$$\xi \sim B(a, b).$$

The associated **probability density function** (PDF) $P$ for the Beta distribution is is given by [6]

$$P(\xi; a, b) = \frac{1}{\text{Beta}(a+1, b+1)} \frac{(\xi+1)^b(1-\xi)^a}{2^{a+b+1}}$$

where

$$\begin{aligned} \text{Beta}(p, q) &:= \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \\ \Gamma(p) &:= \int_0^\infty x^{p-1}e^{-x}\, dx. \end{aligned}$$

Figure 1 shows PDFs for Beta distributions with three different choices of shape parameters.

As discussed in [9], the choice of the system of orthogonal polynomials for the expansion in (4) is dictated by the *Asky scheme*. In short, given a random variable from a known distribution, the Asky scheme provides the choice of a system of orthogonal polynomials based on a correspondence between the weight function of the polynomials and the PDF of the distribution so

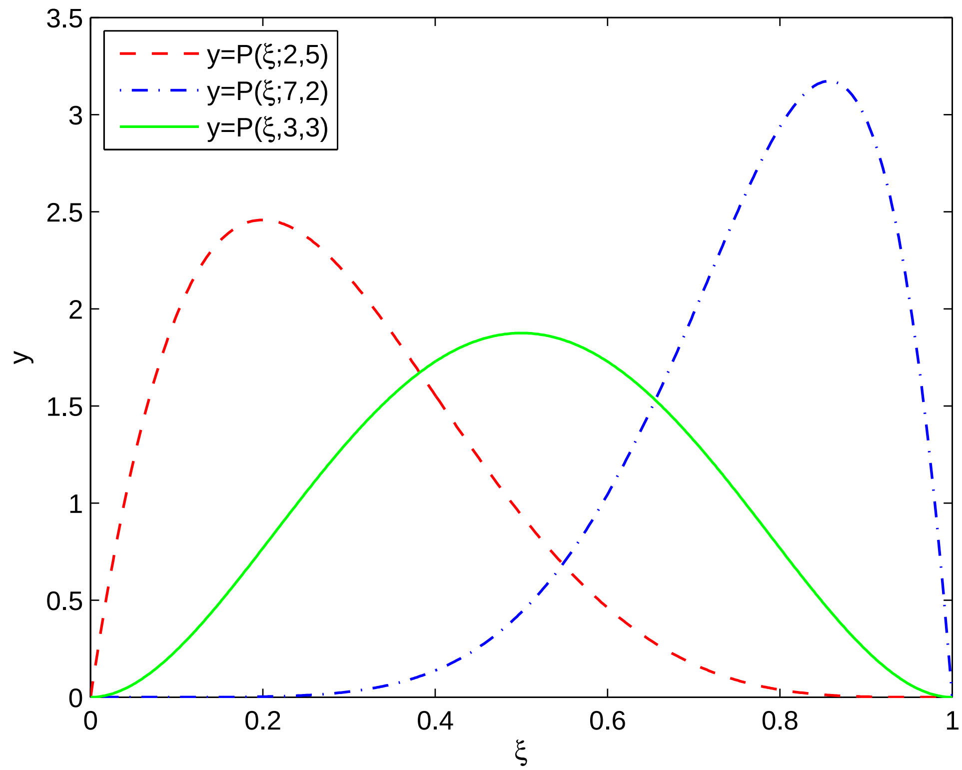


Figure 1: PDFs for example Beta distributions

that the optimal rate of convergence of the GPC expansion is achieved. For $\xi \sim B(a, b)$ the Asky scheme leads to the choice of Jacobi polynomials on $[-1, 1]$ for the system of orthogonal polynomials. These polynomials are characterized by the following weight function, weights, and basic recursion coefficients

$$W(\xi) = (1 - \xi)^a (1 + \xi)^b,$$

$$\gamma_n^2 = \frac{2^{a+b+1}}{2n + a + b + 1} \frac{\Gamma(n + a + 1)\Gamma(n + b + 1)}{n!\Gamma(n + a + b + 1)},$$

and

$$
\begin{aligned}
a_n &= \frac{2(n+a)(n+b)}{(2n+a+b)(2n+a+b+1)}, \\
b_n &= \frac{b^2-a^2}{(2n+a+b)(2n+a+b+2)}, \\
c_n &= \frac{2(n+1)(n+a+b+1)}{(2n+a+b+1)(2n+a+b+2)}.
\end{aligned} \tag{9}
$$

Here are the first three Jacobi polynomials for reference

$$
\begin{aligned}
\varphi_0(\xi) &= 1, \\
\varphi_1(\xi) &= \frac{1}{2}[2(a+1)+(a+b+2)(\xi-1)], \\
\varphi_2(\xi) &= \frac{1}{8}[4(a+1)(a+2)+4(a+b+3)(a+2)(\xi-1) \\
&\quad + (a+b+3)(a+b+4)(\xi-1)^2].
\end{aligned}
$$

We have now finished introducing the basic tools that we shall use throughout the subsequent sections of this paper. Next we discuss the matrices that arise in our later work.

# 3 The Matrices We Love and Why We Love Them

We make use of matrices whenever possible in the following discussions to simplify the presentation and clarify relationships. This section introduces several important matrices appearing throughout the rest of this paper. Several results pertaining to them are also given for later reference.

## 3.1 The Matrix of Basic Recursion Coefficients

Using (9) with $n = 0, 1, \ldots, Q$, we define the **matrix of basic recursion coefficients** of the Jacobi polynomials up to the $Q$th level of recursion

$$M_Q := \begin{bmatrix} b_0 & a_1 & 0 & \ldots & 0 \\ c_0 & b_1 & \ddots & & \vdots \\ 0 & c_1 & \ddots & a_{Q-1} & 0 \\ \vdots & & \ddots & b_{Q-1} & a_Q \\ 0 & \ldots & 0 & c_{Q-1} & b_Q \end{bmatrix} \in \mathbb{R}^{Q+1 \times Q+1}. \tag{10}$$

In the literature, the matrix of basic recursion coefficients is also known as the *Jacobi* matrix [3]. Note that the eigenvalues of $M_Q$ are the roots of the Jacobi polynomial $\varphi_{Q+1}$. This polynomial has $Q+1$ distinct real roots and these are located in the interval $[-1, 1]$.

The matrix of basic recursion coefficients in (10) forms the basis of much of what comes later. Notice that the basic recursion relation (9) expresses the multiplication of $\varphi_n$ by $\xi$ as a linear combination of the Jacobi polynomials of one degree smaller and one degree greater than $\varphi_n$ along with $\varphi_n$ itself. The coefficients of this linear combination are the basic recursion coefficients (9). We want to extend this notion by expressing the multiplication of $\varphi_n$ by $\xi^k$ for $k \in \mathbb{N}$ as a linear combination of Jacobi polynomials. We will call the coefficients in such a linear combination general recursion coefficients.

## 3.2 General Recursion Coefficients

The **general recursion coefficient** of $\varphi_{n+j}$ in the recursion for $\xi^k \varphi_n$ is labeled $C_{n,k}^{(j)}$ and we define this by the relationship

$$\begin{aligned} \xi^k \varphi_n &= C_{n,k}^{(-k)} \varphi_{n-k} + C_{n,k}^{(1-k)} \varphi_{n+(1-k)} + \ldots \\ &\quad + C_{n,k}^{(0)} \varphi_n + \cdots + C_{n,k}^{(k-1)} \varphi_{n+(k-1)} + C_{n,k}^{(k)} \varphi_{n+k} \\ &= \sum_{j=-k}^{k} C_{n,k}^{(j)} \varphi_{n+j}. \end{aligned} \tag{11}$$

To get a firm understanding of what the general recursion coefficients are, let us examine in detail the first couple of steps of the recursive process that leads to (11). We begin with the basic recursion relation from (3)

$$\xi\varphi_n(\xi) = a_n\varphi_{n-1}(\xi) + b_n\varphi_n(\xi) + c_n\varphi_{n+1}(\xi).$$

Comparing this with (11) with $k = 1$, we see that $C_{n,1}^{(-1)} = a_n$, $C_{n,1}^{(0)} = b_n$, and $C_{n,1}^{(1)} = c_n$. To get $C_{n,k}^{(j)}$ for $k = 2$, we multiply both sides of the basic recursion relationship by $\xi$ and obtain

$$\xi^2\varphi_n(\xi) = a_n[\xi\varphi_{n-1}(\xi)] + b_n[\xi\varphi_n(\xi)] + c_n[\xi\varphi_{n+1}(\xi)].$$

Now we apply the basic recursion relation again to each bracketed term and we have

$$\begin{aligned} \xi\varphi_{n-1}(\xi) &= a_{n-1}\varphi_{n-2}(\xi) + b_{n-1}\varphi_{n-1}(\xi) + c_{n-1}\varphi_n(\xi) \\ \xi\varphi_n(\xi) &= a_n\varphi_{n-1}(\xi) + b_n\varphi_n(\xi) + c_n\varphi_{n+1}(\xi) \\ \xi\varphi_{n+1}(\xi) &= a_{n+1}\varphi_n(\xi) + b_{n+1}\varphi_{n+1}(\xi) + c_{n+1}\varphi_{n+2}(\xi). \end{aligned}$$

Substituting these expressions and writing the result in ascending order of the index of the polynomials results in

$$\begin{aligned} \xi^2\varphi_n(\xi) = a_n a_{n-1}\varphi_{n-2}(\xi) + a_n(b_{n-1} + b_n)\varphi_{n-1}(\xi) + (a_n c_{n-1} + b_n^2 + a_{n+1}c_n)\varphi_n(\xi) \\ + c_n(b_n + b_{n+1})\varphi_{n+1}(\xi) + c_n c_{n+1}\varphi_{n+2}(\xi). \end{aligned}$$

From this we can identify the values of the general recursion coefficients for the $k = 2$ case as

$$\begin{aligned} C_{n,2}^{(-2)} &= a_n a_{n-1} \\ C_{n,2}^{(-1)} &= a_n(b_{n-1} + b_n) \\ C_{n,2}^{(0)} &= a_n c_{n-1} + b_n^2 + a_{n+1}c_n \\ C_{n,2}^{(1)} &= c_n(b_n + b_{n+1}) \\ C_{n,2}^{(2)} &= c_n c_{n+1}. \end{aligned}$$

In principle the general recursion coefficients $C_{n,k}^{(j)}$ can be generated by repeated application of the basic recursion relation as was done for the case when $k = 2$. With this approach, however, the calculations quickly become computationally intensive and difficult to manage. We introduce shortly an approach to finding these coefficients that involves simple matrix multiplication. In the definition of the general recursion coefficients $\xi^k\varphi_n$ is expressed as a linear combination of the $k$ polynomials immediately preceding $\varphi_n$ and the $k$ polynomials immediately following $\varphi_n$ along with $\varphi_n$ itself. Also note that the definition of the general recursion coefficient implies that $C_{n,k}^{(j)} = 0$, for $|j| > k$. The following matrix is a convenient way of keeping track of our newly defined general recursion coefficients.

## 3.3 The Matrix of General Recursion Coefficients

Let $k, Q \in \mathbb{N}$ be fixed. We define the **matrix of general recursion coefficients** corresponding to $k$ and $Q$ as

$$W_{Q,k} := \begin{bmatrix} C_{0,k}^{(0)} & C_{1,k}^{(-1)} & \dots & C_{Q,k}^{(-Q)} \\ C_{0,k}^{(1)} & C_{1,k}^{(0)} & \dots & C_{Q,k}^{(1-Q)} \\ \vdots & \vdots & \ddots & \vdots \\ C_{0,k}^{(Q)} & C_{1,k}^{(Q-1)} & \dots & C_{Q,k}^{(0)} \end{bmatrix} \in \mathbb{R}^{Q+1 \times Q+1}. \tag{12}$$

In the following lemma we give a simple way of generating the matrix of general recursion coefficients by powers of the matrix of basic recursion coefficients.

**Lemma 3.1.** *The matrix $W_{Q,k}$ is the top left sub-matrix of $M_{Q+1}^k$. In* MATLAB notation we have

$$W_{Q,k} = M_{Q+1}^k(0 : Q, 0 : Q).$$

**Remark 3.1.** *Notice that in Lemma 3.1 we begin with a matrix $M_{Q+1}$ that is one row and column larger than the size of $W_{Q,k}$. This is only because the element $C_{Q,k}^{(0)}$ in the last row depends on these additional quantities.*

The general recursion coefficients $C_{n,k}^{(j)}$ with $j = -n$ arise later in the paper. Because these coefficients explicitly depend on only two indices, we make the following simplifying definition

$$C_{n,k} := C_{n,k}^{(-n)}. \tag{13}$$

We take advantage of matrix notation once again and define the matrix

$C_Q \in \mathbb{R}^{Q+1\times Q+1}$ of general recursion coefficients $C_{n,k}$ with $0 \le n,k \le Q$ as

$$C_Q := \begin{bmatrix} C_{0,0} & C_{0,1} & \dots & C_{0,Q} \\ C_{1,0} & C_{1,1} & \dots & C_{1,Q} \\ \vdots & \vdots & \ddots & \vdots \\ C_{Q,0} & C_{Q,1} & \dots & C_{Q,Q} \end{bmatrix} = \begin{bmatrix} C_{0,0} & C_{0,1} & C_{0,2} & \dots & C_{0,Q} \\ 0 & C_{1,1} & C_{1,2} & \dots & C_{1,Q} \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & C_{Q,Q} \end{bmatrix}. \tag{14}$$

The upper-triangular shape of $C_Q$ follows from the remark above that $C^{(j)}_{n,k} = 0$ for $|j| > k$.

In the following lemma we show how to obtain the matrix $C_Q$ by iterating through the matrices $W_{Q,j}$, for $j = 0, 1, \dots, Q$ and extracting one column of $C_Q$ at each step of the iterative process.

**Lemma 3.2.** *The $j$th column of $C_Q$ is given by the first row of $W_{Q,j}$. Denoting the $j$th column of $C_Q$ by $C_Q^{[j]}$ and the standard unit (column) vector in $\mathbb{R}^{Q+1}$ as $\hat{e}_1$ we have*

$$\begin{aligned} C_Q^{[j]} &= (W_{Q,j})^T \hat{e}_1 \\ &= (M_Q^j)^T \hat{e}_1. \end{aligned}$$

**Remark 3.2.** *We note that the larger sized $M$ matrix in the definition of $W$ is not required in this context due to the fact that we only use the top row of $W$, and not the last row as referred to in Remark 3.1.*

In the next two lemmas we show some important relationships between general recursion coefficients and inner products of certain orthogonal polynomials. Lemma 3.3 deals with the particular type of general recursion coefficients from (13).

**Lemma 3.3.** *Let $n, k \in \mathbb{N}$ be given. Then*

$$\langle \xi^k, \varphi_n \rangle = \gamma_0^2 C_{n,k}.$$

*Proof.* Let $n, k \in \mathbb{N}$ be given. Then from the definition of $C^{(j)}_{n,k}$ in (11), the definition of Kronecker delta in (1), and the fact that $\varphi_0 \equiv 1$ we have

$$\begin{aligned}
\langle \xi^k, \varphi_n \rangle &= \int_{-1}^{1} \xi^k \varphi_n(\xi)\, dW(\xi) \\
&= \int_{-1}^{1} \sum_{j=-k}^{k} C_{n,k}^{(j)} \varphi_{n+j}(\xi)\, dW(\xi) \\
&= \sum_{j=-k}^{k} C_{n,k}^{(j)} \langle \varphi_{n+j}, \varphi_0 \rangle \\
&= \gamma_0^2 \sum_{j=-k}^{k} C_{n,k}^{(j)} \delta_{n+j,0}. \\
&= \gamma_0^2 C_{n,k}.
\end{aligned}$$

□

Lemma 3.4 below generalizes the previous lemma and shows the connection between inner products and the general recursion coefficients in (11).

**Lemma 3.4.** *Let $Q, k \in \mathbb{N}$ be given. Then for any $n, m \in \mathbb{N}$ with $0 \leq n, m \leq Q$ we have*

$$\langle \xi^k \varphi_m, \varphi_n \rangle = \gamma_m^2 C_{n,k}^{(m-n)}.$$

*Proof.* Fix $Q, k \in \mathbb{N}$. Then we have

$$\begin{aligned}
\langle \xi^k \varphi_m, \varphi_n \rangle &= \int_{-1}^{1} \xi^k \varphi_n(\xi) \varphi_m(\xi)\, dW(\xi) \\
&= \int_{-1}^{1} \sum_{j=-k}^{k} C_{n,k}^{(j)} \varphi_{n+j}(\xi) \varphi_m(\xi)\, dW(\xi) \\
&= \sum_{j=-k}^{k} C_{n,k}^{(j)} \langle \varphi_{n+j}, \varphi_m \rangle \\
&= \gamma_m^2 \sum_{j=-k}^{k} C_{n,k}^{(j)} \delta_{n+j,m} \\
&= \gamma_m^2 C_{n,k}^{(m-n)}.
\end{aligned}$$

□

Note that Lemma 3.3 is a special case of Lemma 3.4 where $m = 0$.

We have now discussed all of the matrices and the associated lemmas that are essential to the methods we propose for solving ODEs in the remainder of this work. Before moving on, we introduce a useful operation on matrices and state an important theorem related to this operation.

## 3.4 The Kronecker Product

Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$. The **Kronecker Product** of $A$ with $B$ is the matrix

$$A \otimes B = \begin{bmatrix} a_{1,1}B & \dots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{m,1}B & \dots & a_{m,n}B \end{bmatrix} \in \mathbb{R}^{mp\times nq}.$$

**Theorem 3.5.** *Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$. If $\lambda$ is an eigenvalue of $A$ with corresponding eigenvector $\mathbf{x}$ and $\mu$ is an eigenvalue of $B$ with corresponding eigenvector $\mathbf{y}$, then $\lambda\mu$ is an eigenvalue of $A\otimes B$ with corresponding eigenvector $\mathbf{x} \otimes \mathbf{y}$.*

A proof of the theorem appears in [5]. We now proceed to apply the tools developed above to solving ordinary differential equations with a random parameter.

# 4 First Order Linear Scalar ODE

The remainder of the paper is concerned with the solution of ODEs involving a random parameter $\xi \sim B(a, b)$. This section deals with the first order linear scalar ODE

$$\dot{u} + \kappa u = g$$

where $\kappa$ is a fixed real number. We refer to the function $g$ on the right hand side of this ODE as the **forcing function**, which we assume is known and depends explicitly on the random parameter and an independent variable $t$. The discussion here is closely related to the work done in [7] for functions with random amplitudes.

Here we use a GPC approach and develop a method of finding the modes of the GPC expansion of a function $u$ that solves the first order linear scalar ODE above. We begin with the initial value problem (IVP)

$$\begin{cases} \dot{u} + \kappa u &= g(t,\xi), \quad t > 0 \\ u(0) &= \alpha \end{cases}. \tag{15}$$

The dependence of the forcing function $g$ on $\xi$ implies a dependence of $u$ on $\xi$ as well. Under the assumptions in the paragraph preceding (4) we expand $u$ and $g$ using Jacobi polynomials in the GPC expansion defined there. We truncate these expansions at the $Q$th degree polynomial to obtain

$$\begin{aligned} u(t,\xi) &\approx u^Q(t,\xi) := \sum_{n=0}^{Q} u_n^Q(t)\varphi_n(\xi), \\ \dot{u}(t,\xi) &\approx \dot{u}^Q(t,\xi) := \sum_{n=0}^{Q} \dot{u}_n^Q(t)\varphi_n(\xi), \\ g(t,\xi) &\approx g^Q(t,\xi) := \sum_{n=0}^{Q} g_n(t)\varphi_n(\xi). \end{aligned} \tag{16}$$

We substitute the truncated expansions (16) into the ODE in (15)

$$\sum_{n=0}^{Q} \dot{u}_n^Q(t)\varphi_n(\xi) + \kappa \sum_{n=0}^{Q} u_n^Q(t)\varphi_n(\xi) = \sum_{n=0}^{Q} g_n(t)\varphi_n(\xi). \tag{17}$$

Formally, $g_n$ is the $n$th mode of the expansion of $g$ as defined in (5), while $u^Q$ is the resulting solution of the ODE system given the level $Q$ trunction in the expansion of random inputs. Thus the GPC coefficients of $u^Q$ may depend on $Q$ (indicated by the superscript). In the case of random forcing, we shall see that the modes of $u^Q$ will not depend on $Q$ and thus we drop the superscript.

We next take advantage of the orthogonality of the polynomial basis of the GPC expansion to eliminate the random variable $\xi$ and obtain a system of ODEs for the modes of the GPC expansions. First we multiply each side of (17) by an arbitrary degree $m$ Jacobi polynomial $\varphi_m(\xi)$ where $0 \le m \le Q$

$$\sum_{n=0}^{Q} \dot{u}_n(t)\varphi_n(\xi)\varphi_m(\xi) + \kappa \sum_{n=0}^{Q} u_n(t)\varphi_n(\xi)\varphi_m(\xi) = \sum_{n=0}^{Q} g_n(t)\varphi_n(\xi)\varphi_m(\xi)$$

We then integrate the sums in the resulting equation term by term with respect to $dW$ and we rewrite the integrals as the equivalent inner products

$$\sum_{n=0}^{Q} \dot{u}_n(t)\langle \varphi_n, \varphi_m \rangle + \kappa \sum_{n=0}^{Q} u_n(t)\langle \varphi_n, \varphi_m \rangle = \sum_{n=0}^{Q} g_n(t)\langle \varphi_n, \varphi_m \rangle.$$

Next we use the definition of the inner product in (2) to arrive at

$$\sum_{n=0}^{Q} \dot{u}_n(t)\delta_{n,m} + \kappa \sum_{n=0}^{Q} u_n(t)\delta_{n,m} = \sum_{n=0}^{Q} g_n(t)\delta_{n,m}.$$

We now let $m = 0, 1, \ldots, Q$ in this equation and obtain a system of decoupled, deterministic ODEs in the variable $t$ for the modes of the GPC expansion. For any $n = 0, 1, \ldots, Q$ we refer to the corresponding ODE involving the mode $u_n$ as the $n$th **modal equation**

$$\dot{u}_n(t) + \kappa u_n(t) = g_n(t). \tag{18}$$

The initial condition corresponding to the $n$th modal equation is found using the definition of the $n$th mode of $u$ in (5). If the initial value in (15) is given as $u(0) = \alpha$ then we have $u_n(0) = \langle \alpha, \varphi_n \rangle / \gamma_n^2 = \alpha \delta_{n,0}$. We then solve these modal initial value problems for $n = 0, 1, \ldots, Q$ and in so doing obtain the modes of $u$. We can also express the decoupled system of modal equations in vector form

$$\dot{\mathbf{w}}_Q(t) + \kappa \mathbf{w}_Q(t) = \mathbf{g}_Q(t)$$

where the vectors here are defined as

$$\mathbf{w}_Q(t) := \begin{bmatrix} u_0(t) \\ \vdots \\ u_Q(t) \end{bmatrix},$$

$$\mathbf{g}_Q(t) := \begin{bmatrix} g_0(t) \\ \vdots \\ g_Q(t) \end{bmatrix}.$$

Recall that the right hand side function $g_n$ in the $n$th modal equation is called the $n$th mode of $g$ in its GPC expansion and is given by

$$g_n(t) := \frac{\langle g, \varphi_n \rangle}{\gamma_n^2}. \tag{19}$$

To solve a given modal equation we must first evaluate the inner product in (19) to get the right hand side. In the absence of an exact analytical solution for the integral that defines $g_n$, we need an accurate and efficient way to approximate this function. One method of approximating $g_n$ is by numerical quadrature. We note that the integration in (19) is with respect to $\xi$ and needs to be computed not once, but at each value of $t$ that $u_n(t)$ is needed. As an alternative for a certain class of functions, we shall explore a method of computing the inner products in (19) indirectly by making further use of the orthogonality of the Jacobi polynomials.

Suppose the function $g$ has a series expansion in a basis of polynomials in $\xi$. In practice we will truncate the full series to obtain a finite sum and arrange in ascending powers of $\xi$ so that the truncation of this expansion of $g$ takes the standard form

$$g(t,\xi) \approx g^N(t,\xi) := \sum_{k=0}^{N} G_k(t)\xi^k. \tag{20}$$

Substituting the approximation in (20) into the inner product in (19) and using Lemma 3.3 gives an approximation $g_n^N$ to the $n$th mode $g_n$ in the GPC expansion of $g$

$$\begin{aligned} g_n(t) \approx g_n^N(t) &:= \frac{\langle \sum_{k=0}^{N} G_k(t)\xi^k, \varphi_n\rangle}{\gamma_n^2} \\ &= \sum_{k=0}^{N} \frac{\langle \xi^k, \varphi_n\rangle}{\gamma_n^2} G_k(t) \\ &= \frac{\gamma_0^2}{\gamma_n^2} \sum_{k=0}^{N} C_{n,k} G_k(t). \end{aligned}$$

Using this approximation for the right hand side in the modal equations (18) gives an approximation $u_n^N$ for each mode $u_n$ by solving the following ODEs, which we shall refer to as the **approximate modal equations**

$$\begin{aligned} \dot{u}_n^N(t) + \kappa u_n^N(t) &= g_n^N(t) \\ &= \frac{\gamma_0^2}{\gamma_n^2} \sum_{k=0}^{N} C_{n,k} G_k(t). \end{aligned}$$

Because $C_{n,k} = C^{(-n)}_{n,k} = 0$ for $n > k$, we have zero right hand side for the approximate modal equations when $n > Q$. Thus we gain no additional information by taking $Q > N$. In the following discussion we assume $N = Q$. The matrix-vector form of the system that results by letting $n = 0, 1, \ldots, Q$ in the approximate modal equations is

$$\dot{\mathbf{w}}_Q(t) + \kappa \mathbf{w}_Q(t) = \gamma_0^2 \Gamma_Q^{-2} C_Q \mathbf{G}_Q(t) \tag{21}$$

where the vectors $\mathbf{w}_Q$, $\mathbf{G}_Q$, and the matrix $\Gamma_Q$ are defined as

$$\mathbf{w}_Q(t) := \begin{bmatrix} u_0^Q(t) \\ \vdots \\ u_Q^Q(t) \end{bmatrix},$$

$$\mathbf{G}_Q(t) := \begin{bmatrix} G_0(t) \\ \vdots \\ G_Q(t) \end{bmatrix},$$

$$\Gamma_Q := \operatorname{diag}(\gamma_0, \ldots, \gamma_Q),$$

and $C_Q$ is the matrix of general recursion coefficients $C_{n,k}$ defined in (14). Once we determine the system of approximate modal equations in (21) we can analytically solve for the vector of approximate modes $\mathbf{w}_Q$ or use a standard ODE solver such as MATLAB's *ode45* to solve for the approximate modes numerically.

**Remark 4.1.** *We emphasize that the matrices $C_Q$ and $\Gamma_Q$ depend only on the choice of orthogonal polynomials (and thus on the distribution and shape parameters describing the random variable). We have separated this from the functional dependence of the forcing term on the random variable described by $\mathbf{G}_Q$.*

**Example 4.1.** *As an example problem we shall solve an (IVP) that represents the case where the forcing function has a random parameter in the form of a frequency*

$$\begin{cases} \dot{u} + u &= f(t, \omega), \quad t > 0 \\ u(0) &= 0 \end{cases} \tag{22}$$

*where the forcing function is $f(t, \omega) = \cos(\omega t)$. We assume the frequencies $\omega$ are distributed uniformly in $[0, \pi]$. Note that this means $\omega$ belongs to the*

*Beta distribution on $[0,\pi]$ with zeros for the shape parameters $a$ and $b$. Using the bijection in (7) we transform to the variable $\xi \sim B(0,0)$, so that $\xi = (2\omega-\pi)/\pi$ and $\omega = \pi(\xi+1)/2$. This gives an IVP equivalent to (22)*

$$\begin{cases} \dot{u}+u &= g(t,\xi), \quad t>0 \\ u(0) &= 0 \end{cases} \tag{23}$$

*where $g(t,\xi)$ in (23) is defined as $f(t,\pi(\xi+1)/2)$. In order to obtain approximate modal equations we expand $g$ as a Taylor series in the variable $\xi$ centered around the point $\xi = 0$ and truncate at $Q$ to obtain*

$$g^Q(t,\xi) = \sum_{k=0}^{Q} \frac{1}{k!} \left. \frac{\partial^k g(t,\xi)}{\partial \xi^k} \right|_{\xi=0} \xi^k.$$

*Comparing this to the standard form in (20), we see that*

$$G_k(t) = \frac{1}{k!} \left. \frac{\partial^k g(t,\xi)}{\partial \xi^k} \right|_{\xi=0}.$$

*For the choice of $g$ in this example we can compute the inner product exactly to get $g_0(t) = \sin(\pi t)/\pi t$. We use this result as the right hand side in the exact zeroth modal equation (18). Using the well-known integrating factor method for linear first order ODEs we obtain an exact solution*

$$u_0(t) = e^{-t} - e^{-t} \int_0^t \frac{e^{\tau}\sin(\pi\tau)}{\pi\tau}\, d\tau.$$

*In fact, since each $G_k$ is a trigonometric function in this example, we can exactly solve each of the approximate modal equations given any $Q$. For computational simplicity, we use* ode15s *with tolerance $10^{-12}$ in a short program to test the accuracy of the approximations from the method above. Figure 2 shows how the approximation $u_0^Q$ compares to the exact solution $u_0$.*

We now look at another example of the first order liner scalar IVP (15).

**Example 4.2.** *This example is similar to the previous one but the forcing function now has a random phase shift $\eta$ uniformly distributed in $[0,\pi]$*

$$\begin{cases} \dot{u}+u &= f(t,\eta), \quad t>0 \\ u(0) &= 0 \end{cases} \tag{24}$$

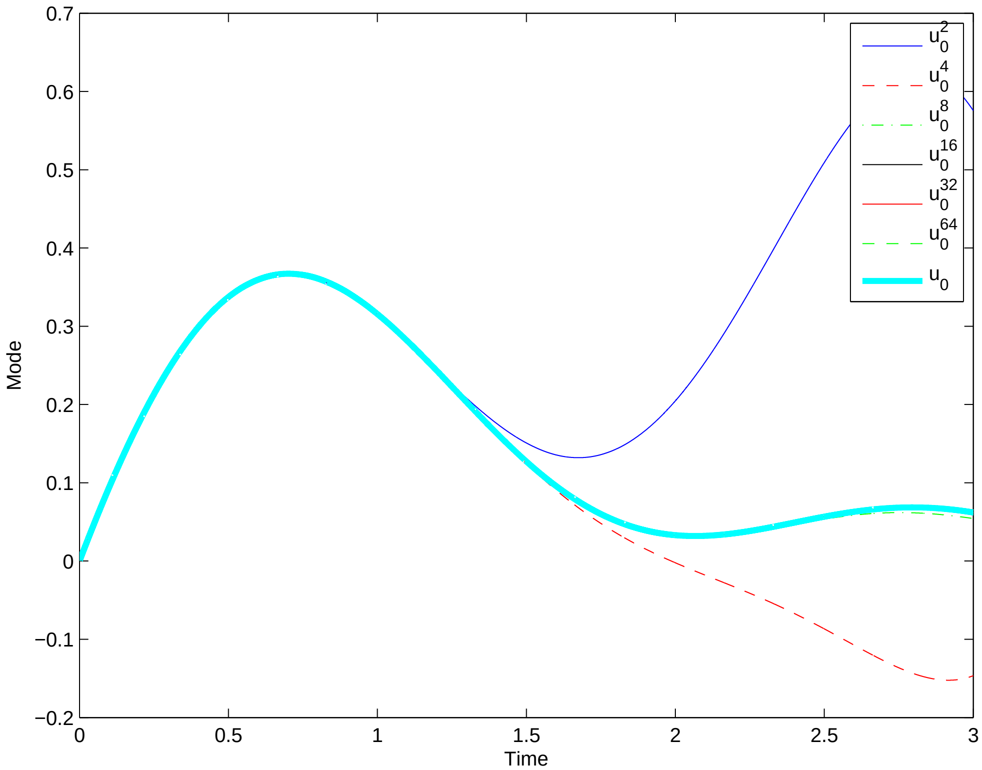


Figure 2: Exact and approximate solutions for random frequency IVP

*where the forcing function takes the form $f(t, \eta) = \cos(\pi t + \eta)$ and $g(t, \xi) = f(t, \pi(\xi + 1)/2)$. For this choice of forcing we compute $g_0(t) = -2\sin(\pi t)/\pi$ and we can again solve the exact zeroth modal equation for $u_0$. Figure 3 shows that for this type of forcing function the convergence appears to be quite rapid and to hold over large values of $t$.*

Figure 4 below shows a plot of the $L^2$ error $||u_0^Q - u_0||_{L^2}$ over the $t$ interval $[0, 5]$ for examples 4.1 and 4.2 . We observe that the rate of convergence for the approximation seems to better in the random phase shift example than for the random frequency example.

This concludes the discussion of first order linear scalar ODEs with random forcing. We now move on to consider analogous systems of ODEs.

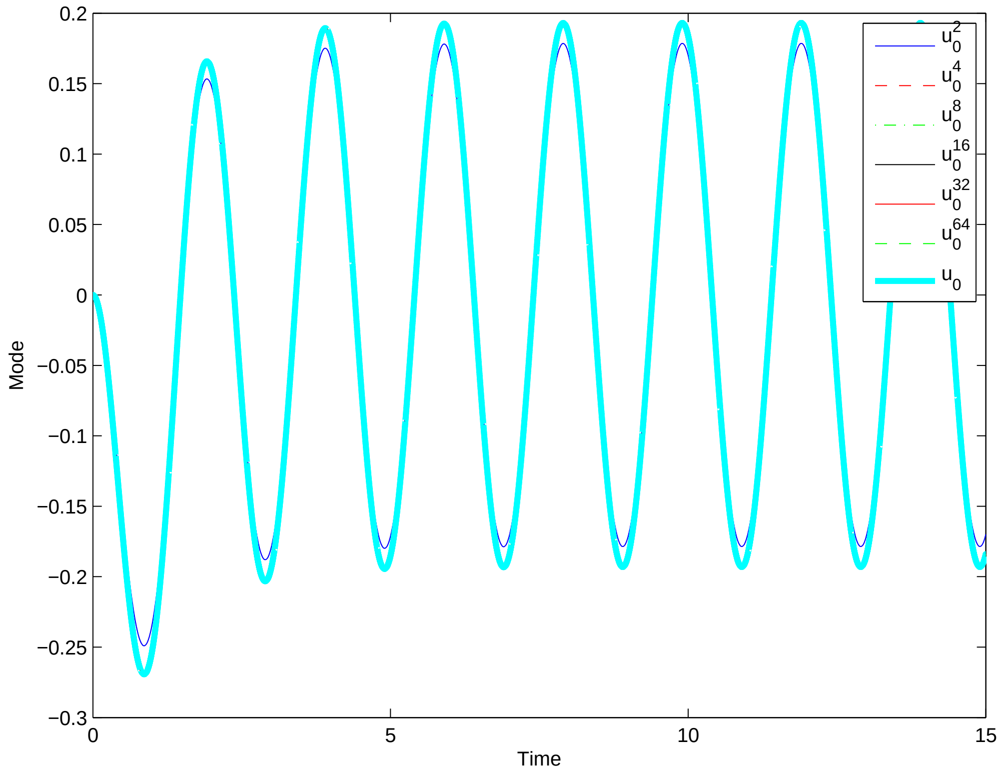


Figure 3: Exact and approximate solutions for random phase shift IVP

# 5 System of ODEs with Random Forcing

Let us now extend the method developed in the previous section for the single scalar ODE with randomness in the forcing function to a system of ODEs with randomness in the forcing function. The ideas presented here extend in a natural way to a system of arbitrary size. We therefore only show the structure for the case of a $2 \times 2$ matrix and we leave the generalization to the larger systems as an exercise for the reader. We begin with the IVP

$$\begin{cases} \dot{\mathbf{w}} + A(t)\mathbf{w} & = \mathbf{f}(t, \xi), \quad t > 0 \\ \mathbf{w}(0) & = \alpha \end{cases} \tag{25}$$

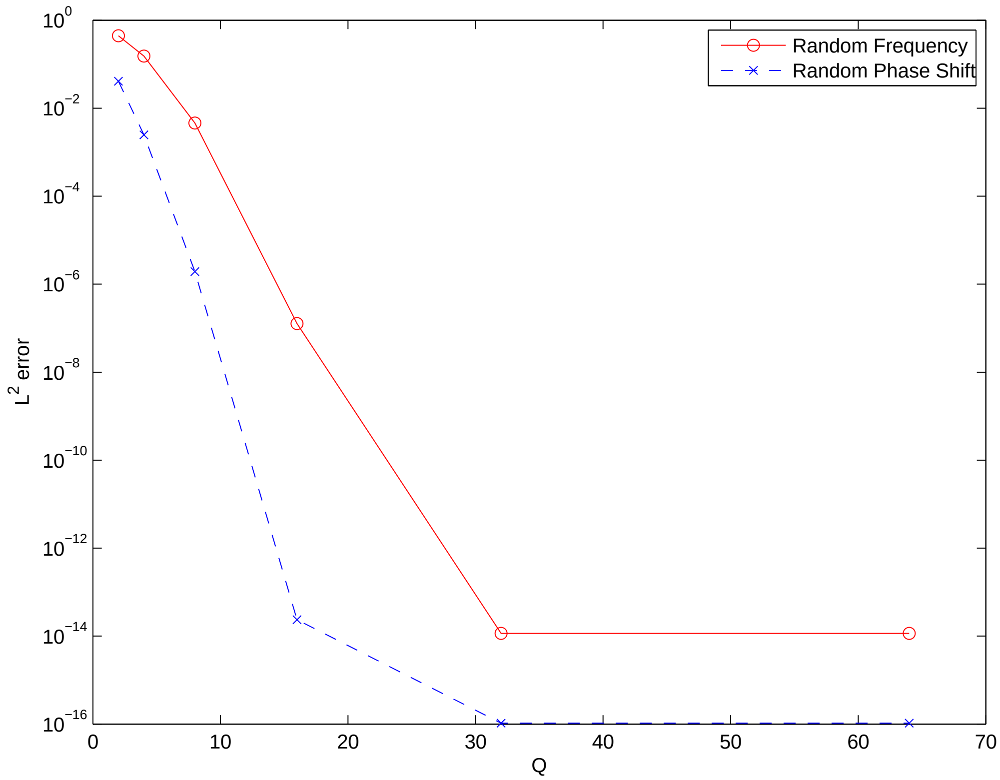


Figure 4: $L^2$ errors for examples 4.1 and 4.2

where the vectors $\mathbf{w}$, $\mathbf{f}$, and $\alpha$ are

$$\mathbf{w} = \begin{bmatrix} u \\ v \end{bmatrix},$$

$$\mathbf{f}(t) = \begin{bmatrix} f(t,\xi) \\ g(t,\xi) \end{bmatrix},$$

$$\alpha = \begin{bmatrix} u(0) \\ v(0) \end{bmatrix}.$$

and where $A$ is the following $2 \times 2$ deterministic matrix

$$A(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}.$$

We assume we can expand the functions $u$, $v$, $f$, and $g$ using GPC. The orthogonality of the Jacobi polynomials leads to the following **modal systems** of ODEs in time for each of the modes

$$\begin{aligned}\dot{u}_n(t) + a(t)u_n(t) + b(t)v_n(t) &= f_n(t)\\ \dot{v}_n(t) + c(t)u_n(t) + d(t)v_n(t) &= g_n(t).\end{aligned} \tag{26}$$

As discussed in the previous section, these modal systems have initial conditions $u_n(0) = u(0)\delta_{n,0}$ and $v_n(0) = v(0)\delta_{n,0}$. We assume here that the elements of $A$, which form the coefficients in the system in (26) and the modes $f_n$ and $g_n$ are all continuous on some common neighborhood of the point $t = 0$ where the initial condition is given in (25). This allows us to use standard existence and uniqueness theory to guarantee a solution to the $n$th modal system in this neighborhood. The right hand side functions in (26) are given as in (19) by

$$\begin{aligned}f_n(t) &= \frac{\langle f, \varphi_n\rangle}{\gamma_n^2}\\ g_n(t) &= \frac{\langle g, \varphi_n\rangle}{\gamma_n^2}.\end{aligned}$$

Note that each of the systems in (26) is decoupled in the sense that for each $n$ the system can be solved independently from systems for other values. The matrix form of (26) is

$$\dot{\mathbf{y}}_n + A\mathbf{y}_n = \mathbf{F}_n$$

where

$$\begin{aligned}\mathbf{y}_n &:= \begin{bmatrix} u_n(t) \\ v_n(t) \end{bmatrix},\\ \mathbf{F}_n &:= \begin{bmatrix} f_n(t) \\ g_n(t) \end{bmatrix}.\end{aligned}$$

If the integrals in the expressions for $f_n$ and $g_n$ above cannot be computed exactly, they can be approximated by using numerical quadrature for example or, when applicable, by techniques analogous to those developed in the previous section, i.e., using expansions analogous to those in (20) and the matrix $C_Q$ in (14) to compute approximations to the resulting integrals. The approximate modal system can then be solved for $n = 0, 1, \ldots, Q$ using a standard ODE solver or analytically if possible.

This ends our exploration into ODEs with random forcing functions. We now move on to consider systems of ODEs with randomness in the coefficients of the system. We first consider a system with a deterministic matrix multiplied by a random variable $\xi A(t)$, which we then generalize an to affine function of the random variable times a deterministic matrix $(r\xi + m)A(t)$. Next we consider the case with a deterministic matrix multiplied by an arbitrary power of a random variable $\xi^k A(t)$. Finally we consider the case where the system involves a possibly nonlinear function of a random variable $\xi$ and a deterministic system matrix $R(A(t), \xi)$.

# 6 System of ODEs with Linear Randomness in the Coefficients

Thus far we have considered systems of ODEs where the randomness appeared explicitly only in the right hand side forcing function. We now consider systems with randomness in the coefficients on the left hand side. We assume that the forcing function and the system matrix depend only on $t$, i.e., that they are deterministic. We discuss the details for the case where the system matrix is $2 \times 2$, which leads to an obvious generalization for larger systems.

## 6.1 Systems with $\xi A(t)$ on the LHS

First we consider the IVP

$$\begin{cases} \dot{\mathbf{w}} + \xi A(t)\mathbf{w} &= \mathbf{f}(t), \quad t > 0 \\ \mathbf{w}(0) &= \alpha \end{cases} \tag{27}$$

where $\mathbf{w}$, $\mathbf{f}$, $\alpha$, and $A$ are

$$\begin{aligned}\mathbf{w} &= \begin{bmatrix} u \\ v \end{bmatrix}, \\ \mathbf{f}(t) &= \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}, \\ \alpha &= \begin{bmatrix} u(0) \\ v(0) \end{bmatrix}, \\ A(t) &= \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}. \end{aligned} \tag{28}$$

Writing (27) as a system gives

$$\begin{aligned}\dot{u} + a\xi u + b\xi v &= f \\ \dot{v} + c\xi u + d\xi v &= g. \end{aligned} \tag{29}$$

We expand $u$ and $v$ using GPC and truncate the expansions at $Q$. We then proceed along the same lines as in the previous sections to use properties of the system of orthogonal polynomials to generate a system of ODEs involving the modes of these expansions. After substituting the truncated GPC expansion for $u$ and $v$ into (29), we multiply both equations by a general $\varphi_m$ where $0 \leq m \leq Q$ to get

$$\begin{aligned}\sum_{n=0}^{Q} \dot{u}_n \varphi_n \varphi_m + a \sum_{n=0}^{Q} u_n \xi \varphi_n \varphi_m + b \sum_{n=0}^{Q} v_n \xi \varphi_n \varphi_m &= f \varphi_m \\ \sum_{n=0}^{Q} \dot{v}_n \varphi_n \varphi_m + c \sum_{n=0}^{Q} u_n \xi \varphi_n \varphi_m + d \sum_{n=0}^{Q} v_n \xi \varphi_n \varphi_m &= g \varphi_m. \end{aligned} \tag{30}$$

Integrating the equations in (30) with respect to the random variable $\xi$ and rewriting the integrals as inner products gives

$$\begin{aligned}\sum_{n=0}^{Q} \dot{u}_n \langle \varphi_n, \varphi_m \rangle + a \sum_{n=0}^{Q} u_n \langle \xi \varphi_n, \varphi_m \rangle + b \sum_{n=0}^{Q} v_n \langle \xi \varphi_n, \varphi_m \rangle &= \langle f, \varphi_m \rangle \\ \sum_{n=0}^{Q} \dot{v}_n \langle \varphi_n, \varphi_m \rangle + c \sum_{n=0}^{Q} u_n \langle \xi \varphi_n, \varphi_m \rangle + d \sum_{n=0}^{Q} v_n \langle \xi \varphi_n, \varphi_m \rangle &= \langle g, \varphi_m \rangle. \end{aligned} \tag{31}$$

Because $f$ and $g$ depend only on the variable $t$ we have

$$\begin{aligned}\langle f, \varphi_m\rangle &= \langle 1, \varphi_m\rangle f = \gamma_m^2 \delta_{m,0} f \\ \langle g, \varphi_m\rangle &= \langle 1, \varphi_m\rangle g = \gamma_m^2 \delta_{m,0} g.\end{aligned} \tag{32}$$

In order to deal with the inner products $\langle \xi\varphi_n, \varphi_m\rangle$ in the summations in (31) we apply Lemma 3.4 with $k = 1$, which gives

$$\langle \xi\varphi_n, \varphi_m\rangle = \gamma_m^2 C_{n,1}^{(m-n)}. \tag{33}$$

We substitute (32) and (33) into (31) and we divide both sides of the resulting equations by the common factor $\gamma_m^2$. This gives us the following

$$\begin{aligned}\sum_{n=0}^{Q} \dot{u}_n \delta_{n,m} + a \sum_{n=0}^{Q} u_n C_{n,1}^{(m-n)} + b \sum_{n=0}^{Q} v_n C_{n,1}^{(m-n)} &= \delta_{m,0} f \\ \sum_{n=0}^{Q} \dot{v}_n \delta_{n,m} + c \sum_{n=0}^{Q} u_n C_{n,1}^{(m-n)} + d \sum_{n=0}^{Q} v_n C_{n,1}^{(m-n)} &= \delta_{m,0} g.\end{aligned}$$

Letting $m = 0, 1, \ldots, Q$ gives a $2(Q+1) \times 2(Q+1)$ coupled system of deterministic ODEs for the modes of $u$ and $v$

$$\dot{\mathbf{y}}_{\mathbf{Q}} + (A \otimes M_Q)\mathbf{y}_{\mathbf{Q}} = \mathbf{F}_Q \tag{34}$$

where $M_Q = W_{Q,1}$ is the Jacobi matrix of basic recursion coefficients in (10), $\otimes$ is the Kronecker product, and

$$\mathbf{y}_{\mathbf{Q}} := \begin{bmatrix} u_0 \\ \vdots \\ u_Q \\ v_0 \\ \vdots \\ v_Q \end{bmatrix},$$

$$\mathbf{F}_Q(t) := \begin{bmatrix} f(t) \\ 0 \\ \vdots \\ 0 \\ g(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{35}$$

The initial conditions for the modal system in (34) are $u_n(0) = u(0)\delta_{n,0}$ and $v_n(0) = v(0)\delta_{n,0}$.

We now wish to generalize the result in (34) to one for the case where we have an affine function of the random variable multiplied by a deterministic system matrix.

## 6.2 Systems with $(r\xi + m)A(t)$ on the LHS

We take advantage of the discussion above to state an approximate modal system analogous to (34) for the following IVP involving an affine function of the random variable $r\xi + m$ multiplied by a deterministic matrix $A(t)$, which may be viewed as a shifting and scaling of the random variable $\xi$ as mentioned in the Preliminaries section in (8)

$$\begin{cases} \dot{\mathbf{w}} + (r\xi + m)A(t)\mathbf{w} & = \mathbf{f}(t), \quad t > 0 \\ \mathbf{w}(0) & = \alpha \end{cases} \tag{36}$$

where $\mathbf{w}$, $\mathbf{f}$, $\alpha$, and $A$ are as in (28). The result above in (34) gives the following approximate modal systems corresponding to the intital value problem in (36)

$$\dot{\mathbf{y}}_{\mathbf{Q}} + A \otimes (rM_Q + mI_Q)\mathbf{y}_{\mathbf{Q}} = \mathbf{F}_Q. \tag{37}$$

Where $\mathbf{y}_{\mathbf{Q}}$ and $\mathbf{F}_Q$ are the same as in (35), $I_Q \in \mathbb{R}^{Q+1 \times Q+1}$ is the identity matrix, and the initial conditions are the same as those stated for (34). We remark here that in modal system (37) we can clearly distinguish the roles of the system in the matrix $A$, the distribution in the matrix $M_Q$, and the scaling and shifting in the parameters $r$ and $m$.

Before moving on to state the main result for a system ODEs with a nonlinear function of a random variable and a deterministic matrix we need one more result.

## 6.3 Systems with $\xi^k A(t)$ on the LHS

We now consider a system of ODEs with randomness in the coefficients of the left hand side of the system in the form of a power of a random variable times a deterministic system matrix. We begin with the following IVP

$$\begin{cases} \dot{\mathbf{w}} + \xi^k A(t)\mathbf{w} & = \mathbf{f}(t), \quad t > 0 \\ \mathbf{w}(0) & = \alpha \end{cases} \tag{38}$$

where $\mathbf{w}$, $\mathbf{f}$, $\alpha$, and $A$ are as in (28). Writing (38) as a system gives

$$\begin{aligned} \dot{u} + a\xi^k u + b\xi^k v &= f \\ \dot{v} + c\xi^k u + d\xi^k v &= g. \end{aligned} \tag{39}$$

Following the same approach as in the previous section gives an approximate modal system analogous to that in (31), with $\xi^k$ replacing $\xi$

$$\begin{aligned} \sum_{n=0}^{Q} \dot{u}_n \langle \varphi_n, \varphi_m \rangle + a \sum_{n=0}^{Q} u_n \langle \xi^k \varphi_n, \varphi_m \rangle + b \sum_{n=0}^{Q} v_n \langle \xi^k \varphi_n, \varphi_m \rangle &= \langle f, \varphi_m \rangle \\ \sum_{n=0}^{Q} \dot{v}_n \langle \varphi_n, \varphi_m \rangle + c \sum_{n=0}^{Q} u_n \langle \xi^k \varphi_n, \varphi_m \rangle + d \sum_{n=0}^{Q} v_n \langle \xi^k \varphi_n, \varphi_m \rangle &= \langle g, \varphi_m \rangle. \end{aligned} \tag{40}$$

The only new difficulty here is the appearance of the quantity $\langle \xi^k \varphi_n, \varphi_m \rangle$ in (40). Applying Lemma 3.4 gives

$$\langle \xi^k \varphi_n, \varphi_m \rangle = \gamma_m^2 C_{n,k}^{(m-n)}. \tag{41}$$

Substituting (32) and (41) into (40) and dividing by $\gamma_m^2$ gives

$$\begin{aligned} \sum_{n=0}^{Q} \dot{u}_n \delta_{n,m} + a \sum_{n=0}^{Q} u_n C_{n,k}^{(m-n)} + b \sum_{n=0}^{Q} v_n C_{n,k}^{(m-n)} &= \delta_{m,0} f \\ \sum_{n=0}^{Q} \dot{v}_n \delta_{n,m} + c \sum_{n=0}^{Q} u_n C_{n,k}^{(m-n)} + d \sum_{n=0}^{Q} v_n C_{n,k}^{(m-n)} &= \delta_{m,0} g. \end{aligned}$$

Again letting $m = 0, 1, \ldots, Q$ gives the following modal system

$$\dot{\mathbf{y}}_{\mathbf{Q}} + (A \otimes W_{Q,k})\mathbf{y}_{\mathbf{Q}} = \mathbf{F}_Q. \tag{42}$$

Where $\mathbf{y}_{\mathbf{Q}}$ and $\mathbf{F}_Q$ are as in (35), $W_{Q,k}$ is the matrix of general recursion coefficients in (12), and $\otimes$ is the Kronecker product. This deterministic system of ODEs can be solved analytically or approximately with a standard solver. Note that since $M_Q = W_{Q,1}$ we have (34) as a special case of (42) with $k = 1$.

### 6.4 Overall Qualitative Behavior

Consider the solutions to the approximate modal system from (37), which approximate the solutions to the random system

$$\dot{\mathbf{w}} + (r\xi + m)A(t)\mathbf{w} = \mathbf{f}(t)$$

given in (36). We make some remarks here about how these approximations compare qualitatively to solutions of the non-random system

$$\dot{\mathbf{w}} + A(t)\mathbf{w} = \mathbf{f}(t).$$

From the theory of orthogonal polynomials we have that the eigenvalues of the matrix of basic recursion coefficients $M_Q$ in (10) are the roots of the Jacobi polynomial $\varphi_{Q+1}$. This polynomial has $Q + 1$ distinct real roots in the interval $[-1, 1]$, which means that if $\lambda$ is an eigenvalue of $M_Q$, then $\lambda \in [-1, 1]$. Thus if we have $m > r > 0$, then $rM_Q + mI_Q$ has $Q + 1$ distinct real eigenvalues in $[m - r, m + r]$. If we scale the random variable $\xi$ appropriately by $r$ and $m$ so that $rM_Q + mI_Q$ has positive eigenvalues, then by Theorem 3.5 we conclude that the signs of the eigenvalues of the Kronecker product $A \otimes (rM_Q + mI_Q)$ in the approximate modal systems in (37) are the same as those of the matrix $A$. Furthermore, we can conclude from Theorem 3.5 that if $A$ has only real eigenvalues then $A \otimes (rM_Q + mI_Q)$ also has only real eigenvalues, and if $A$ has only complex eigenvalues, then $A \otimes (rM_Q + mI_Q)$ does as well. This qualitative analysis shows that that the stability and overall behavior of solutions to the non-random system are preserved in the approximations to $u$ and $v$. We do note that multiple oscillation modes can give the appearance of decaying amplitudes on small time intervals, thus the short-time behavior of the random solution may be somewhat different than that of the deterministic one.

We now move on to apply the results of this section to the more general situation involving a possibly nonlinear function as part of the system.

# 7 System of ODEs with General Randomness on LHS

We are now ready to state a method for approximating the solution to a system of ODEs involving a possibly nonlinear function of a random variable

$\xi \sim B(a, b)$ and a deterministic system matrix. We state results for the case where the system matrix $A$ is $2 \times 2$, $A\colon [0, \infty) \to \mathbb{C}^{2\times 2}$, and leave the extension to larger systems for the reader.

Consider the following IVP

$$\begin{cases} \dot{\mathbf{w}} + R(A(t), \xi)\mathbf{w} &= \mathbf{f}(t), \quad t > 0 \\ \mathbf{w}(0) &= \alpha \end{cases} \tag{43}$$

where the vectors $\mathbf{w}$ and $\mathbf{f}$, and the matrix $A$ are defined above in (28), and where

$$R\colon \mathbb{C}^{2\times 2} \times [-1, 1] \to \mathbb{C}^{2\times 2} \tag{44}$$

is a possibly nonlinear function. We assume that we can expand $R$ as its Taylor series in the random variable $\xi$ about the point $\xi = 0$, which we truncate at $N$ to get

$$R^N(t, \xi) := \sum_{k=0}^{N} A_k(t)\xi^k \tag{45}$$

where $A_k\colon [0, \infty) \to \mathbb{C}^{2\times 2}$ for $k = 0, 1, \ldots, N$ are the Taylor coefficients of the expansion of $R$. This assumption leads to the following coupled system of deterministic ODEs for the approximate modes of $u$ and $v$, which is analogous to that in (42)

$$\dot{\mathbf{y}}_{\mathbf{Q}} + \left( \sum_{k=0}^{N} A_k(t) \otimes W_{Q,k} \right) \mathbf{y}_{\mathbf{Q}} = \mathbf{F}_Q. \tag{46}$$

By solving the system in (46) we obtain approximations for the modes $u_0, \ldots, u_Q$ and $v_0, \ldots, v_Q$. We note here that we do not directly address the case when expansions of the form given in (45) cannot be used nor do we discuss issues of convergence of such expansions. For a discussion of these issues we refer the reader to [5].

We conclude this section with an example of the ideas presented above in which we find approximate solutions to the predator-prey model with randomness in the coefficients of the system.

**Example 7.1.** *We consider the linearized homogeneous predator-prey model used to model the change from equilibrium of coexisting populations of a prey species u and a predator species v. The model is given by the following IVP*

$$\begin{cases} \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} + \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad t > 0 \\ \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \end{cases} \tag{47}$$

*where the constants b and c in the coefficient matrix of the system are both positive, and $\alpha$ and $\beta$ are given initial values for u and v respectively. We introduce random noise into the system in (47) through multiplication of a shifted and scaled random variable to get the following*

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} + (r\xi + m) \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

*This is a problem of the type (36) that was discussed in the previous section. Applying the method developed there gives the following system of approximate modal equations*

$$\dot{\mathbf{y}}_{\mathbf{Q}} + A \otimes (rM_Q + mI_Q)\mathbf{y}_{\mathbf{Q}} = \mathbf{0}$$

*where*

$$A = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix}, \qquad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

*We take $r = 0.01$, $m = 1$, $\xi \sim B(0,0)$, $\alpha = \beta = 100$, and let $b = 0.5 = c$. Solving the resulting system gives approximations for the modes of u and v. Figure 5 shows a plot of $u_0^Q$ for $Q = 4$ with confidence envelopes for the expected value of the prey species, showing plus or minus one standard deviation using the formula for variance in (6). Figure 6 displays a similar plot for the predator species.*

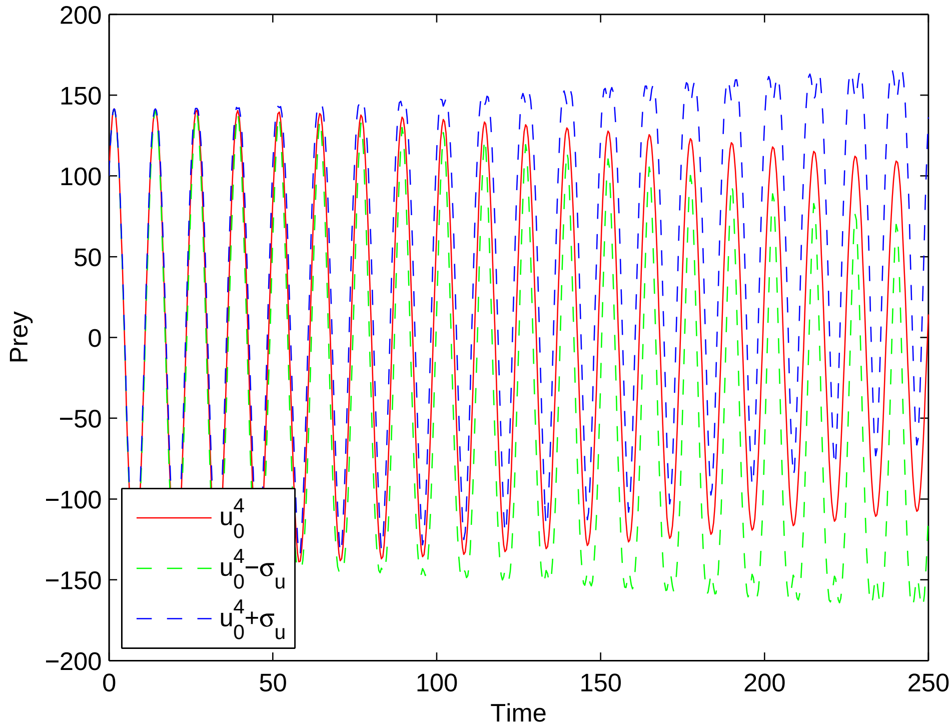


Figure 5: $u_0^4$ and confidence envelopes for the random predator-prey model

*For comparison, we solve the non-random predator-prey system and find the deterministic solutions $u$ and $v$. These solutions are plotted in Figure 7. We see that because the matrix $A$ has complex eigenvalues, the system exhibits stable oscillations and the species continue to coexist over time.*

*Figures 5 and 6 seem to indicate that the addition of random noise to the system in this case results in a change in the qualitative behavior of our approximate solutions, since the populations of both species seem to be decaying to a fixed value over time. Figure 8 shows this over a longer period of time. We argue that this is merely caused by the presence of multiple modes and that further time integration would show the amplitude returning to the initial level.*

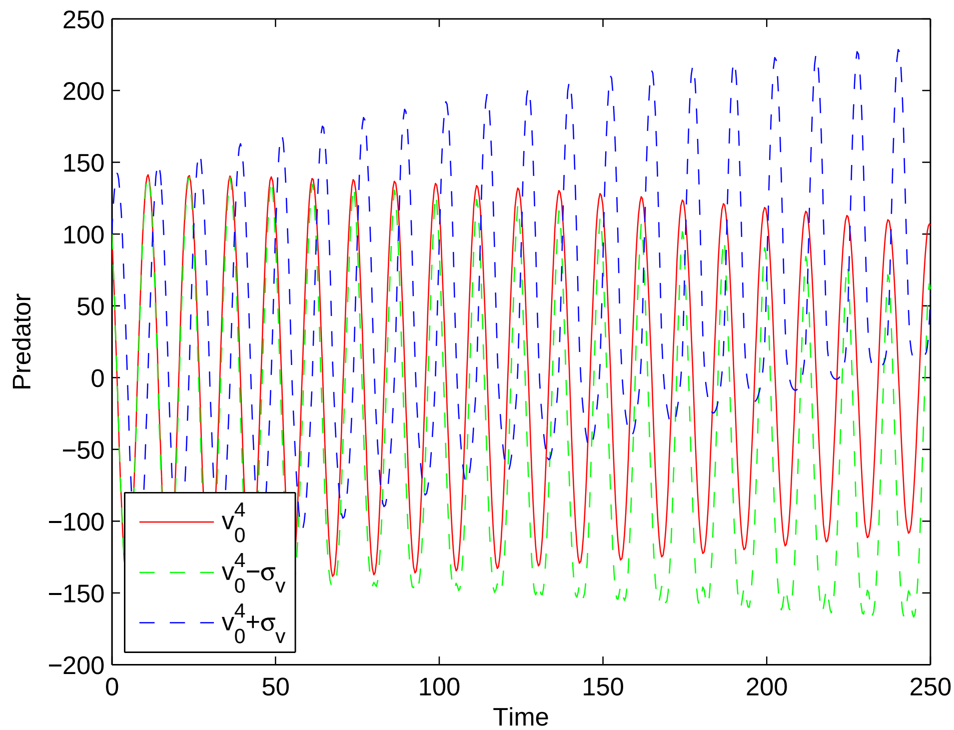


Figure 6: $v_0^Q$ and confidence envelopes for the random predator-prey model

# 8 Conclusions and Future Work

In our work here we have developed methods for solving models with ODEs involving random parameters in the forcing function and in the coefficients of the system. The main tool that we have used in this development has been the Generalized Polynomial Chaos (GPC) approach, which begins with an expansion of the functions in the ODEs and leads to systems of deterministic ODEs for the modes of these expansions. By using this approach one can incorporate uncertainty in a mathematical model by modeling uncertain parameters with random variables and proceed to solve the resulting deterministic systems for the expansion modes and obtain approximate solutions to the model.

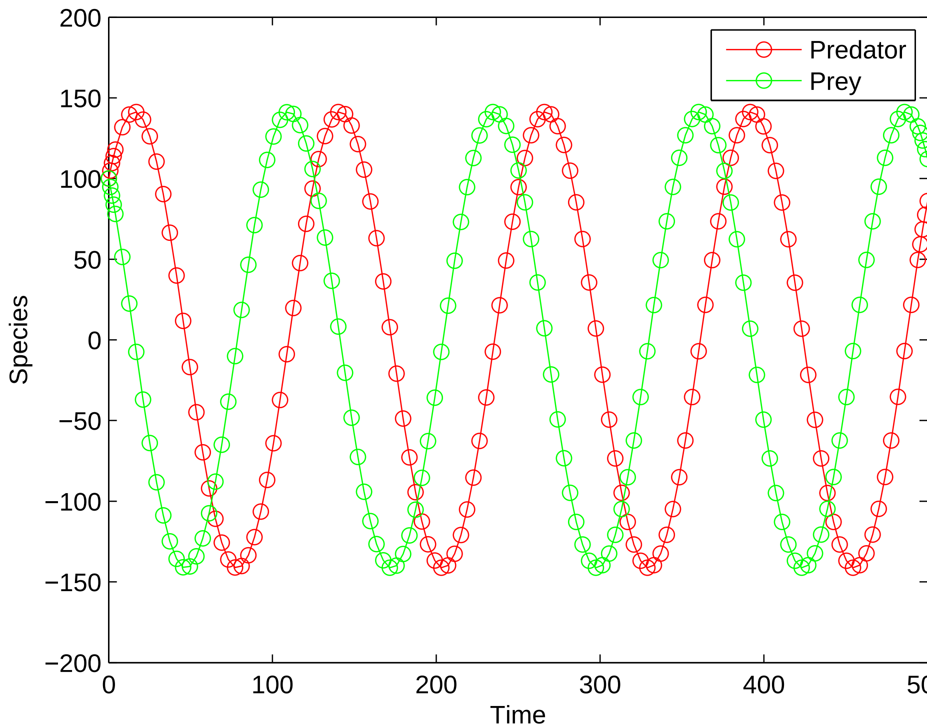


Figure 7: Solutions to the non-random predator-prey models

Future research possibilities related to the ideas covered in this paper include extensions to Partial Differential Equations (PDEs) involving random parameters. This would include using the Finite Element Method or the Method of Lines approach to solving elliptic Boundary Value Problems with a random parameter. We also hope to connect our work here with other research in numerical solutions to Maxwell's system of PDEs in Debye media. In particular, we wish to apply the methods developed in the current paper to the systems of ODEs that arise in the application of the Yee scheme to Maxwell's equations.

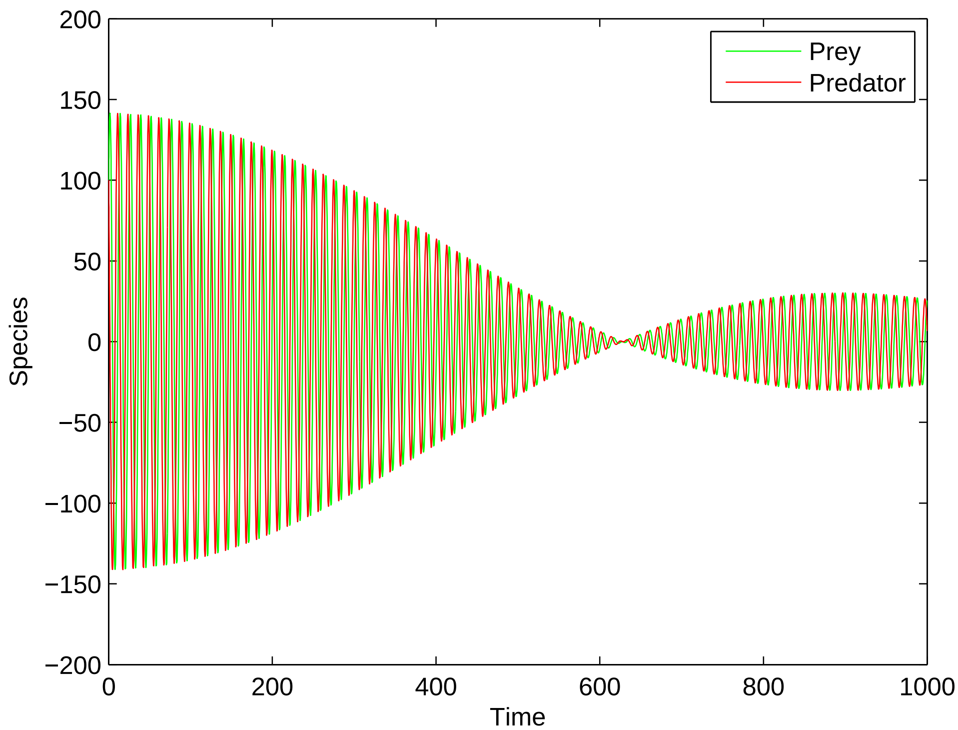


Figure 8: Solutions to the random predator-prey models

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